Étale Cohomology

An Essay for CASM

by

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§1. Introduction

1.1 Pre-requisites

This essay has been written as part of the one year Certificate of Advanced Study in Mathematics (CASM) course at Cambridge University which coincides with Part III of the Mathematical Tripos. The starting point is, of necessity, roughly that reached in the lectures which in this particular year did not include much in the way of schemes and sheaves, nor, in the case of the author, much in the way of algebraic number theory.

Thus the frontiers of the subject can safely rest undisturbed by the contents of this essay. Rather it has been written with a reader in mind corresponding roughly to the author at the start of the enterprise. That is someone who is interested to find out what all the fuss was with the French algebraic geometers in the 1960s but is in need of some fairly elementary background to map out the abstractions involved and with any luck to avoid drowning in the "rising sea".

1.2 Approach

In line with the acknowledged wisdom of starting with the work of the masters, the motivation for the inclusion of topics in this essay has been taken from the lexically challenging SGA $4\frac{1}{2}$ by Deligne (D1). This has the added bonus of being in French thus contributing to the general romance (and opportunity for error) in the undertaking.

Where this assumes a degree of mathematical sophistication not possessed by this writer (depressingly often and frequently without it being realised at first) an attempt has been made to include sufficient background material both as catharsis for the author and as a checklist for the reader. This is mostly taken from standard works on Algebraic Geometry (e.g. Hartshorne, H1) and Étale Cohomology (e.g. Milne, M1 and M2).

To make productive use of étale cohomology in, for example, obtaining the results in number theory that were part of the initial motivation for its development, one needs to take advantage of the fact that it allows the construction of a cohomology theory with coefficients in a field of characteristic zero for varieties over fields of arbitrary characteristic. This is the l-adic cohomology.

The mathematically ambitious might rush to this point and establish the foundations of "Proper base change", "Duality" etc. before moving on to some significant results in the theory. This is not the approach of this essay. Rather it aims to recap the basic ideas from algebraic geometry and cohomology before outlining some of the generalisations for which we have mainly Grothendieck to thank and Deligne to explain. Wherever a diagram comes to mind an attempt has been made to include one, often in lieu of a more closely constructed analysis.

There is a lot of machinery involved (which might be second nature to the more expert practitioner but their attention will surely be elsewhere by now) but Deligne kindly furnishes us with some results as well, and in particular for curves. Philosophically he suggests that curves are the key, and the expert by use of the appropriate "devissages" (which might be translated as "tricks" but this could be considered to lack the necessary gravitas) can reduce many problems to their study. This essay gets to the point of setting out some of these results, with minor diversions to pick up the Brauer and Picard groups and a little bit of Kummer theory along the way. Of necessity, time has to be spent wandering through some of the basics of Galois cohomology as this provides the link to certain of the calculations in low degrees.

1.3 Scope

This essay starts with foundational material on algebraic geometry and cohomology leading up to the definition of sheaf cohomology on a suitable category. Without establishing this common ground, what follows will not make much sense to the uninitiated. Experts can of course skip this, pausing perhaps only to admire some of the diagrams.

At this point some of Grothendieck's (many) ideas are introduced to establish a generalised notion of topology leading to the definition of étale cohomology as a particular instance of a sheaf cohomology.

As a base camp for the assault on the cohomology of curves, some results on the vanishing of the Brauer group (whose construction is carefully explained) and its implications for cohomology are established. In addition some results about the Picard group and the Kummer sequence are stated for later use.

Finally results about the cohomology of curves with coefficients in a constant sheaf are established with a little help from some properties of Jacobians and algebraic varieties. This can be viewed as the starting point for dealing with higher dimensions (through skilful reduction to the case of curves by fibering, say) and venturing beyond the safety of torsion sheaves to consider the l-adic case. But not in this essay.

1.4 Background

For an algebraic variety (as reviewed below) defined over the complex numbers there is an induced topology available to allow the application of the methods of algebraic topology. Where it is defined over an arbitrary algebraically closed field then the only available construction of open sets is via the Zariski topology which is too coarse for the methods of algebraic topology to yield much useful information.

In the 1940s Weil identified that the existence of a suitable cohomology theory over finite fields would enable some key conjectures in number theory to be proved but was unable to construct it (see Freitag, F1 for the requirements of a suitable cohomology theory).

To give a flavour of the issues involved here is a bald statement:

If X is a suitable variety over some finite field $\mathbb{F}_q$ then the following zeta function on X can be defined:

$$\zeta(X, s) = exp\left(\sum_{r=1}^{\infty} N_r \frac{(q^{-s})^r}{r}\right)$$

where N_r is the number of points of X over $\mathbb{F}_{q^r}$. Then Weil made a number of conjectures about the zeta function (e.g. it is a rational function of q^{-s} and the corresponding polynomials have degrees that relate to topological properties of X) and in particular that its zeros and poles satisfy the equivalent of the Riemann hypothesis (see for example Weil, W2).

In considering some of the issues involved, Weil was led to re-examine the foundations of algebraic geometry (Weil, W1). Building on the work of Serre amongst others, Grothendieck worked on an approach *"suggested to me by the connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand and the classification of unramified coverings of a variety on the other"* (Grothendieck, G2).

A decade or so after Weil's conjectures, Grothendieck introduced an étale topology for schemes which led to the development of the theory of étale cohomology (based on sheaves and their derived functors). Typically this involved extensions to the notion of a point (boring old closed points acquired generic siblings and geometric cousins who dropped in from outside the space) and of topological open sets (by coverings which were no longer point sets, only somewhat open and not really all that topological). In due course these ideas led to the proof of the Weil conjectures (finally by Deligne in the 1970s) and much else besides.

These abstract, elegant, category-theoretical developments turned out to be very powerful. They not only provide the expert with beautiful generalisations (and rather poetic imagery) but also with insights to tackle concrete problems (e.g. representation theory of Deligne and Lusztig). Grothendieck's ideas would seem now to be firmly established as the basis for doing algebraic geometry.

§2. Algebraic Geometry Foundations

(The treatment here broadly follows Hartshorne, H1). This section recaps the relationships between polynomial rings and varieties and provides a formal definition of sheaves on a topological space. This is a prelude to introducing affine schemes based on more general rings which leads to a definition of a scheme and the associated morphisms.

2.1 Algebraic sets and varieties

Recall that if $\mathsf{A} = k[x_1, \cdots, x_n]$ is a polynomial ring over an algebraically closed field and $\mathbf{A^n}$ is affine n-space over k, then if $\mathsf{T} \subseteq \mathsf{A}$, the zero set of T is defined as

$$Z(\mathsf{T}) = \{P \in \mathbf{A^n} | f(P) = 0 \, \forall f \in \mathsf{T}\}$$

and a subset Y of $\mathbf{A^n}$ is an algebraic set if there exists a $\mathsf{T} \subseteq \mathsf{A}$ such that $Y = Z(\mathsf{T})$. The Zariski topology on $\mathbf{A^n}$ is defined by taking open sets to be the complements of the algebraic sets. An affine variety is then an irreducible closed subset of $\mathbf{A^n}$.

Hilbert's famous Nullstellensatz establishes the following relationships:

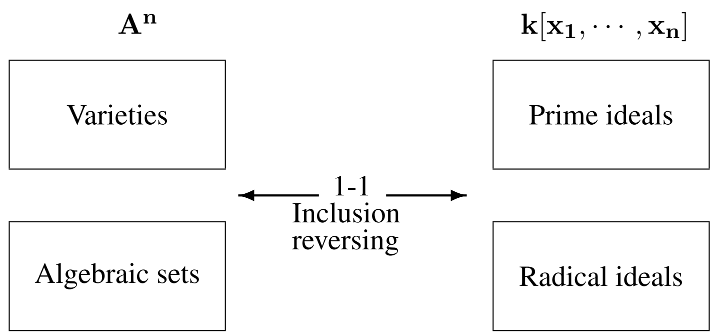

Figure 1: Nullstellensatz

In particular, a maximal ideal in A corresponds to a minimal irreducible closed subset in $\mathbf{A^n}$ i.e. to a point. If $I(Y)$ is the ideal corresponding to an algebraic set Y, then the affine coordinate ring $\mathsf{A}(Y)$ of Y is the quotient $\mathsf{A} \,/\, I(Y)$.

With a few refinements there is a similar set of relationships in projective n-space over k:

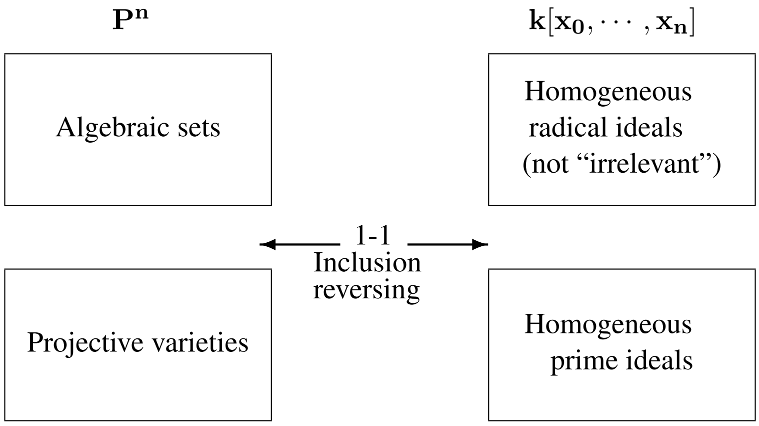

Figure 2: Projective Nullstellensatz

Projective n-space has an open covering by affine n-spaces.

2.2 Sheaves on a topological space

Let X be a topological space. A presheaf $\mathscr{F}$ of Abelian groups on X consists of the data

a) for every open subset $U \subseteq X$, an Abelian group $\mathscr{F}(U)$ and

b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of Abelian groups

$$\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V),$$

subject to the conditions

0) $\mathscr{F}(\emptyset) = 0$ where $\emptyset$ is the empty set,

1) ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and

2) if $W \subseteq V \subseteq U$ are three open subsets then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rr, "\rho_{UV}"] \arrow[dr, "\rho_{UW}"'] & & \mathscr{F}(V) \arrow[dl, "\rho_{VW}"] \\
& \mathscr{F}(W) &
\end{tikzcd}$$

A presheaf of rings, sets or any other category is defined by replacing "Abelian group" with the appropriate object.

$\mathscr{F}(U)$ is referred to as the sections of $\mathscr{F}$ over the open set U. (It is usually a set of some kind, hence the use of the plural). The ρ_{UV} are called restriction maps so we can write $s_{|V}$ instead of $\rho_{UV}(s)$, if $s \in \mathscr{F}(U)$.

A sheaf on X is a presheaf that satisfies the following additional conditions:

3) If U is an open set, $\{V_i\}$ an open covering of U and $s \in \mathscr{F}(U)$ is an element such that

$$s_{|V_i} = 0 \text{ for all } i$$

Then $s = 0$

4) If U is an open set, $\{V_i\}$ an open covering of U and we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i, j

$$s_{i_{|V_i \cap V_j}} = s_{j_{|V_i \cap V_j}}$$

then there is an element $s \in \mathscr{F}(U)$ such that

$$s_{|V_i} = s_i \text{ for each } i$$

(That is if there are local data which are compatible, they "patch together" to form something in $\mathscr{F}(U)$).

2.3 Schemes

Schemes generalise the notion of an algebraic variety by extending the relationship outlined above with a polynomial ring. Any ring A is now considered and "points" will correspond to prime ideals not just the maximal ones. It also means that we can consider rings over fields that are not algebraically closed.

2.3.1 Spectrum of a ring

With any ring A we associate a topological space Spec(A) consisting of all the prime ideals in A. If $\mathfrak{a}$ is any ideal of A then the subset $V(\mathfrak{a}) \subseteq$ Spec(A) is the set of all prime ideals that contain $\mathfrak{a}$.

The topology on Spec(A) comes from taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets. This topology has a basis of open sets $\{D_f$ for $f \in \mathsf{A}\}$ where

$$D_f = \{\mathfrak{p} \in \text{Spec}(\mathsf{A}): f \notin \mathfrak{p}\}$$

To complete the picture we need to define a sheaf of rings $\mathscr{O}$ on Spec(A):

If U is an open set in Spec(A), then if $s \in \mathscr{O}(U)$ is a section we want s to map the elements of U (prime ideals) into the corresponding localisation of A. i.e. we want s to be a function

$$s : U \to \coprod_{\mathfrak{p} \in U} \mathsf{A}_{\mathfrak{p}} \text{ such that } s(\mathfrak{p}) \in \mathsf{A}_{\mathfrak{p}}$$

and such that for each $\mathfrak{p}$ there is a neighbourhood $V_{\mathfrak{p},s}$ contained in U and elements a, f in A such that for each $\mathfrak{q}$ in $V_{\mathfrak{p},s}$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f}$. In summary:

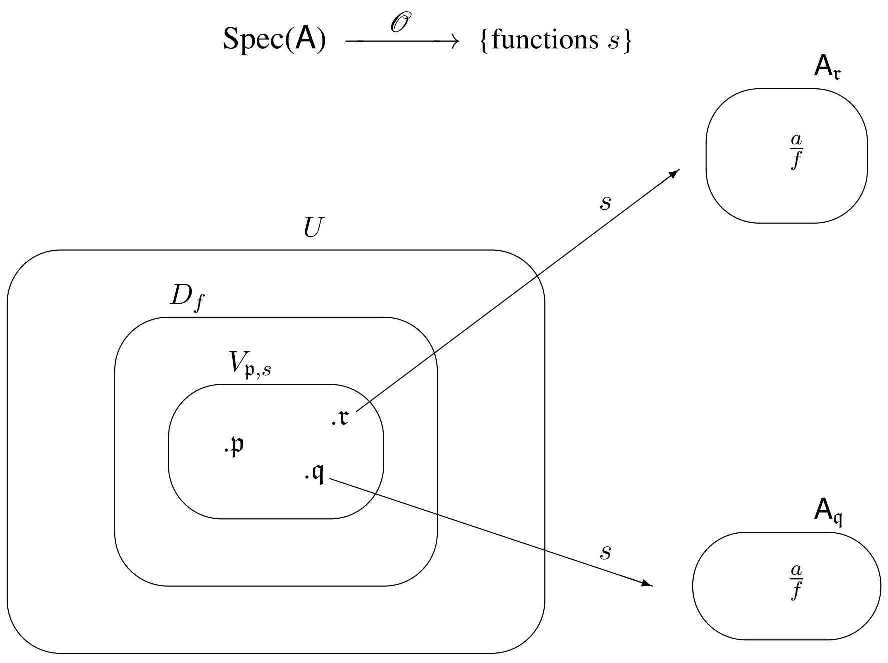

Figure 3: Section of sheaf of rings

The *spectrum* of A is then defined as the topological space Spec(A) equipped with the sheaf of rings $\mathscr{O}$. To complete the agricultural constructions, we define the stalk $\mathscr{O}_{\mathfrak{p}}$ to be the germ of sections of $\mathscr{O}$ at $\mathfrak{p}$ as an equivalence class of pairs of open sets and functions as follows:

$$\mathscr{O}_{\mathfrak{p}} = \{[< U, s >]/\sim \text{ where } \mathfrak{p} \in U \text{ and } s \in \mathscr{O}(U)\} \text{ and } \sim \text{ is defined by}$$

$$[< U, s >] \sim [< V, t >] \Leftrightarrow \exists\, W \subseteq U \cap V : s_{|W} = t_{|W}$$

It can be shown that $\mathscr{O}_{\mathfrak{p}}$ is a local ring isomorphic to $\mathsf{A}_{\mathfrak{p}}$ and for any $f \in \mathsf{A}$, the ring $\mathscr{O}(D_f)$ is isomorphic to the localised ring A_f (H1 Chapter 2, Proposition 2.2).

A *ringed space* is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X. It is a *locally ringed space* if in addition for each point $\mathrm{P} \in X$ the stalk $\mathscr{O}_{X,\mathrm{P}}$ is a local ring. Thus the spectrum of a ring is a locally ringed space.

An *affine scheme* can then be defined as a locally ringed space which is isomorphic to the spectrum of some ring.

A *scheme* is then a locally ringed space $(X, \mathscr{O}_X)$ in which every point has an open neighbourhood U such that $(U, \mathscr{O}_{X_{|U}})$ is an affine scheme.

2.4 Morphisms

An X-scheme (or a scheme over X) is a scheme Y and a morphism $Y \longrightarrow X$. This begs the question as to the nature of a morphism of schemes. As schemes have structure at different levels it is perhaps not surprising that we need two maps to define a morphism that respects these.

A morphism of schemes can be viewed as a morphism of locally ringed spaces and comprises two maps $(f, f^{\#})$ whose actions are best illustrated by the following diagram which uses the above notations:

$$
\begin{tikzcd}
Y \arrow[rr, "f", "cts"'] & & X & \text{Top. spaces} \\
\cup & & \cup & \\
f^{-1}(V) \arrow[d, "\mathscr{O}_Y"] & & V \arrow[ll, "f^{-1}"'] \arrow[d, "\mathscr{O}_X"] \arrow[dll, "f_*\mathscr{O}_Y"'] & \text{sheaves} \\
\mathscr{O}_Y(f^{-1}(V)) \arrow[d, "\mathrm{P} \in f^{-1}(V)"] & & \mathscr{O}_X(V) \arrow[ll, "f^{\#}(V)"', "hom"] \arrow[d, "f(\mathrm{P}) \in (V)"] & \text{rings} \\
\varinjlim_V \mathscr{O}_Y(f^{-1}(V)) \arrow[d] & & \varinjlim_V \mathscr{O}_X(V) \arrow[d, phantom, "="] & \\
\mathscr{O}_{Y,\mathrm{P}} & & \mathscr{O}_{X,f(\mathrm{P})} \arrow[ll, "f^{\#}_{\mathrm{P}}"', "local\ hom"] & \text{stalks (local rings)} \\
\cup & & \cup & \\
\mathfrak{m}_{Y,\mathrm{P}} \arrow[rr, "f^{\#^{-1}}_{\mathrm{P}}", "="'] & & \mathfrak{m}_{X,f(\mathrm{P})} & \text{unique (maximal ideals)}
\end{tikzcd}
$$

Figure 4: Morphism of schemes

The final local homomorphism on the stalks requires that the maximal ideal is pulled back to the maximal ideal.

§3. Cohomology Foundations

This section recaps the basic constructions from algebraic topology which are briefly illustrated with simplicial homology and De Rham cohomology. A little more time is spent setting up Galois cohomology (as this is used later in support of étale cohomology) and illustrating it with the proof of Hilbert's Theorem 90.

The notions are then extended to a more general setting of suitable categories (Abelian with enough injectives) via derived functor cohomology and applied to the global sections functor of an Abelian sheaf to define sheaf cohomology (Grothendieck showed that this is a suitable category). This is the level of abstraction that will be required for the étale case.

3.1 Algebraic topology

The basic requirement for constructing a cohomology is the identification of a cochain complex. i.e. a sequence of Abelian groups and homomorphisms

$$\dashrightarrow C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} C^{n+2} \dashrightarrow \qquad\qquad \text{s.t. } d^2 = 0$$

The cohomology groups can then be defined as

$$H^n(C^*, d) = \frac{\text{Ker } d : (C^n \to C^{n+1}) \text{ cocycles}}{\text{Im } d : (C^{n-1} \to C^n) \text{ coboundaries}}$$

A most important general result is the extension of a short exact sequence of complexes (i.e. a short exact sequence between the components)

$$0 \to A^* \to B^* \to C^* \to 0$$

to a long exact sequence of cohomology

$$\cdots \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots$$

with a natural map $\delta^i : H^i(C^*) \to H^{i+1}(A^*)$.

There are several different methods of associating (co)homology groups with a topological space and in 1952 Eilenberg and Steenrod (Eilenberg, E1) stated the axioms (exactness, homotopy, excision and dimension) which would ensure the essential uniqueness of a theory.

3.2 Singular homology for topological spaces

In the category of topological spaces we can construct singular homology from a chain complex comprising the (huge!) free Abelian groups $C_n(X))$ generated by maps of n-simplices into a space X with the homomorphisms being the boundary maps on the generators. The corresponding cochain complex then comprises the groups

$$C^n(X) = \text{Hom}(C_n(X), \mathbb{Z})$$

with coboundary homomorphisms d given by

$$d(\psi)\sigma = \psi(d\sigma)$$

where $\psi \in \text{Hom}(C_n(X), \mathbb{Z})$ and σ maps an $(n+1)$-simplex into X.

3.3 De Rham cohomology for smooth manifolds

In the category of smooth manifolds we denote the space of smooth r-forms on a manifold M by $\Omega^r(M)$. Then exterior differentiation gives us a map

$$d : \Omega^r(M) \to \Omega^{r+1}(M)$$

and we can construct the r^{th} De Rham cohomology

$$H^r_{\text{DR}}(M) := \frac{\text{Closed } r\text{-forms}}{\text{Exact } r\text{-forms}}$$

3.4 Group (Galois) cohomology

*3.4.1 Theorem: **G**-modules and boundary operator*

A group module A is an Abelian group equipped with the action of a group G that respects its structure. i.e. a map

$$G \times A \to A$$

$$(\sigma, a) \mapsto \sigma a$$

such that $\tau(\sigma a) = \sigma(\tau a)$ and $1a = a$
and in addition $\sigma(a + b) = \sigma a + \sigma b \;\; \forall\, \sigma, \tau \in G$ and $a, b \in A$.
In short, A is a G-module.
Strictly, if we write Λ for the algebra $\mathbb{Z}[G]$ then A is a (left) Λ-module with

$$\left(\sum n_\sigma \sigma\right) a = \sum n_\sigma(\sigma a),$$

$C^n(G, A)$ is then defined as the Abelian group of maps of G^n to A.
The coboundary homomorphism $d : C^n(G, A) \to C^{n+1}(G, A)$ is defined by the formula (see Serre, S1 §2.2)

$$\begin{aligned}(df)(g_1, \cdots, g_{n+1}) &= g_1 f(g_2, \cdots, g_{n+1}) \\ &+ \textstyle\sum_{i=1}^n (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \cdots, g_n)\end{aligned}$$

and $d^2 = 0$ as required.

Thus, for example, we have

$H^0(G, A) = A^G$ (the group of elements of A fixed by G)

and in $H^1(G, A)$ the 1-cocycles correspond to maps $f : G \to A$ where

$f(g_1 g_2) = g_1.f(g_2) + f(g_1) \quad \forall\, g_1, g_2 \in G$ ("crossed homomorphisms")

and the coboundaries correspond to maps of the form

$f(g) = g.a - a$ for some $a \in A$.

To illustrate the fact that some of these groups can actually be calculated, we have the following example:

3.4.2 Theorem: (Generalised) Hilbert Theorem 90

If L is a finite Galois extension of a field K then $H^1(\text{Gal}(L/K, L^*) = 0$

Proof (based on lecture notes on Elliptic Curves)

Write G for Gal(L/K). Suppose f is a 1-cocycle and $\sigma, \tau \in G$. Then in the multiplicative group L^* we have, from the above definition,

$$f(\sigma\tau) = f(\sigma).(\sigma(f(\tau)))$$

and in particular

$$\sigma(f(\tau)^{-1}) = (\sigma(f(\tau)))^{-1} = f(\sigma).f(\sigma\tau)^{-1}.$$

Now consider the element of G given by the Poincaré series $\sum_{\tau\in G} f(\tau)^{-1}\tau$.

Since distinct automorphisms in G are linearly independent, this is not 0. That is there is a y in L^* such that

$$x = \sum_{\tau\in G} f(\tau)^{-1}\tau(y) \quad \neq 0$$

Then if we apply σ in G we have (using the above identity)

$$\sigma(x) = f(\sigma) \sum_{\tau\in G} f(\sigma\tau)^{-1}\sigma\tau(y) = f(\sigma)x$$

Or $f(\sigma) = \sigma(x).x^{-1}$ and hence f is a co-boundary.

(Hilbert's original result can be recovered by considering the case where G is cyclic with σ as a generator and $x \in L^*$ has Norm = 1, then $\exists\ y \in K^*$ such that $x = y/\sigma(y)$).

3.4.3 Profinite group actions

If G is a topological group and A is given the discrete topology we can consider the sub-category of G-modules where the action is continuous. $C^n(G, A)$ is then defined as the set of all continuous maps of G^n to A.

A topological group which is the projective limit of finite groups, each given the discrete topology, is called a *profinite group*. Such a group is compact and totally disconnected (Serre, S1).

An important example is the case where we have a field k and K a Galois extension of k (not necessarily finite) with G the Galois group Gal(K/k). (Serre, S2 Chapter 10, §3).

G is a profinite group equal to the projective limit of the Galois groups $G(K'/k)$ where K' runs through the set of finite Galois sub-extensions of K. If A is a G-module then it is a topological G-module if

$$\forall\ a \in A, \{\sigma \in G\ |\sigma a = a\}\ \text{ is an open subset of } G$$

i.e. $A = \bigcup A^N$ where N runs through the open normal subgroups of G. We can then define:

$$H^q(G, A) = \varinjlim H^q(G/N, A^N)$$

In particular if K is the algebraic closure of k, then Hilbert's Theorem 90 gives us that

$$H^1(Gal(K/k),\ K^*) = 0.$$

3.4.4 Non-Abelian G-modules

If A is not Abelian then the above process does not lead to cohomology groups. However all is not lost and it is possible to construct $H^0(G, A)$ and $H^1(G, A)$ in such a way as to agree with the above when A is Abelian. In the Non-Abelian case the H^i are pointed sets rather than groups.

3.5 Categorical cohomology

3.5.1 Derived functor cohomology

3.5.1.1 Suitable categories for cohomology

The above examples of cohomology rely to some extent on the characteristics of the category under consideration. A very general category can be defined which has sufficient properties (such as the notion of kernel, cokernel etc. so we can have things like exact sequences) and where the morphisms have the structure of an Abelian group. Such a category is an Abelian category (Milne, M1 Chapter III section 1).

A complex A^* in an Abelian category is a collection of objects (now not necessarily Abelian groups) A^i and morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. We can still define the i^{th} cohomology object $H^i(A^*)$ of the complex A^* as ker $d^i/\text{im } d^{i-1}$.

A key component in the definition of a cohomology for such a category is the notion of a derived functor which in turn requires the construction of injective resolutions. This imposes some additional constraints on a suitable category.

3.5.1.2 Injective resolution and derived functors

(The following definitions and constructions follow Hartshorne, H1 Chapter III §1).

If $\mathfrak{A}$ is an Abelian category and A a fixed object, then the functor

Hom$(., A) : B \to$ Hom(B, A) is always left exact.

There may be objects where we can go a bit further: An object I of $\mathfrak{A}$ is *injective* if Hom$(., I)$ is exact.

An *injective resolution* of an object A of $\mathfrak{A}$ is a complex I^* defined for $i \geq 0$ together with a morphism $\varepsilon : A \to I^0$ such that I^i is an injective object of $\mathfrak{A}$ for each i, and such that the sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \cdots$$

Is exact.

If the category has *enough injectives* then every object has an injective resolution. Now if $\mathfrak{A}$ has enough injectives and $F : \mathfrak{A} \to \mathfrak{B}$ is a covariant left exact functor to another Abelian category, then the *right derived functors* $R^iF,\ i \geq 0$ of F are constructed as follows:

For each object A in $\mathfrak{A}$ fix an injective resolution as above and write it as

$$0 \to A \to I^*$$

and apply F to the complex I^* to get the complex $F(I^*)$, then take cohomology to define

$$R^iF(A) := H^i(F(I^*))$$

It can be shown that this construction is independent of the injective resolution chosen and that R^iF is an additive functor (from $\mathfrak{A}$ to $\mathfrak{B}$) with $F \cong R^0F$ and with long exact sequences following from short exact sequences etc.

3.5.2 Sheaf cohomology

If $\mathfrak{g}$ is a category (that is assumed to have a final object X) and $\mathfrak{Ab}(\mathfrak{g})$ is the category of Abelian sheaves on $\mathfrak{g}$ (i.e. functors into the category $\mathfrak{Ab}$ of Abelian groups), then a key result of Grothendieck showed that $\mathfrak{Ab}(\mathfrak{g})$ is an Abelian category with enough injectives.

Let $\Gamma = \Gamma(X, .) : \mathfrak{Ab}(\mathfrak{g}) \to \mathfrak{Ab}$ be the global sections functor and $\mathscr{F} \in \mathfrak{Ab}(\mathfrak{g})$, then for each $i \geq 0$, the i^{th} *derived functor cohomology group of* $\mathscr{F}$ is defined as

$$H^i(X, \mathscr{F}) = R^i\Gamma(\mathscr{F})$$

This can be shown to define a cohomology theory that satisfies the required axioms. In particular, if $\mathscr{I}$ is injective then

$$H^i(X, \mathscr{I}) = 0 \text{ for all } i > 0.$$

§4. Étale developments

("The word apparently refers to the appearance of the sea at high tide under a full moon in certain types of weather" - Mumford, M5 p175)

There are two objectives:-

1. To extend the concept of topology. The Zariski topology produces very large open sets with the results (amongst others) that:
   (a) If X has complex dimension n then $H^i(X, \mathscr{F}) = 0$ for all $i > n$ although X has real dimension $2n$ and should have higher cohomology.
   (b) If X is an irreducible topological space and $\mathscr{A}$ is a constant sheaf then
   $$H^i(X, \mathscr{A}) = 0 \text{ for } i > 0 \text{ (Grothendieck's Theorem).}$$
2. To extend the base field to include those not necessarily algebraically closed.

This section shows how the idea of an open set is abstracted in the definition of Grothendieck topologies, which uses the notion of a "scheme over a base scheme" with the link being provided by a morphism of schemes. To ensure that additional nice properties are available, the allowable morphisms are restricted to étale morphisms, which are defined.

Thus an étale topology is arrived at with the corresponding definition of an étale site. Sheaves on such a site are defined together with some examples that will be needed later. This opens up the construction of étale cohomology as a sheaf cohomology.

Comparison theorems are not really tackled in this essay other than to note the equivalence of étale and Galois cohomology when the base scheme is the spectrum of a field.

This section concludes with a brief interlude to collect together the various examples that can reasonably be considered to represent a point.

4.1 Sheaves on a category - Grothendieck topology

(This generally follows the treatment given by Deligne (D1 Chapter I)).

We have been working with sheaves defined on the category of open sets of a space X (where the morphisms are inclusion maps) where the available definition is via the Zariski topology.

The general idea is to abstract the open sets to objects in a more general category and inclusions to morphisms in this category. These can be used to define a notion of topology on the category (the Grothendieck topology). In due course by specialising the category somewhat (to schemes étale over a base scheme) this leads to non-trivial cohomologies.

First we need a construction for handling local properties.

4.1.1 Sieves

Let X be a topological space, $f : X \to \mathbb{R}$ a real-valued function on X.

The continuity of f is a local property:- if f is continuous on every sufficiently small open set of X, f is continuous on the whole of X.

To formalise the notion of a local property a few definitions can be introduced:

One says that a collection $\underline{\mathfrak{U}}$ of open sets in X is a *sieve* if for all $U \in \underline{\mathfrak{U}}$ and open $V \subset U$ one has $V \in \underline{\mathfrak{U}}$.

One says that a sieve is a *covering* if the union of all the open sets belonging to the sieve is equal to X.

Given a family $\{U_i\}$ of open sets of X, the sieve generated by $\{U_i\}$ is by definition the collection of open sets U of X such that $U \subset U_i$ for some U_i.

One says that a property $\mathsf{P}(U)$ defined for each open set U of X is *local* if for every covering sieve $\underline{\mathfrak{U}}$ of each open set $U \subset X$,

$$\mathsf{P}(U)\text{is true} \Leftrightarrow \mathsf{P}(V) \text{ is true} \ \forall\ V \subset \underline{\mathfrak{U}}$$

(e.g. " f is continuous on U").

This idea can be extended to more general categories as follows:

Let $\mathfrak{g}$ be a category and U an object of $\mathfrak{g}$. Then a sieve on U is a subset $\underline{\mathfrak{U}}$ of morphisms of $\mathfrak{g}$ with codomain U such that if $\phi : V \to U$ belongs to $\underline{\mathfrak{U}}$ and if $\psi : V \to W$ is a morphism in $\mathfrak{g}$ then $\phi \circ \psi : W \to U$ belongs to $\underline{\mathfrak{U}}$.

If $\{\phi_i : U_i \to U\}$ is a family of morphisms, the sieve generated by the U_i is by definition the set of morphisms $\phi : V \to U$ which factorise through one of the ϕ_i.

If $\underline{\mathfrak{U}}$ is a sieve on U and if $\phi : V \to U$ is a morphism, the restriction $\underline{\mathfrak{U}}_V$ of $\underline{\mathfrak{U}}$ to V is by definition the sieve on V consisting of the morphisms $\psi : V \to U$ such that $\phi \circ \psi : W \to U$ belong to $\underline{\mathfrak{U}}$.

4.1.2 Grothendieck topology

The specification of a Grothendieck topology on $\mathfrak{g}$ consists of the specification for each object U of $\mathfrak{g}$ of a set $C(U)$ of sieves on U, called covering sieves, such that the following axioms are satisfied:

1. The sieve generated by the identity on U is covering.

2. If $\underline{\mathfrak{U}}$ is a covering sieve on U and if $V \to U$ is a morphism, the sieve $\underline{\mathfrak{U}}_V$ is covering. Equivalently if $\underline{\mathfrak{U}} = \{\phi_i : U_i \to U\}$ then the fibre products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}$ is a covering for V. (So fibre products perform the role that intersections play for point sets).

3. A locally covering sieve is covering. In other words, if $\underline{\mathfrak{U}}$ is a covering sieve on U and if $\underline{\mathfrak{U}'}$ is a sieve on U such that for all $V \to U$ belonging to $\underline{\mathfrak{U}}$ the sieve $\underline{\mathfrak{U}'}_V$ is covering, then $\underline{\mathfrak{U}'}$ is covering.

A category equipped with a Grothendieck topology is called a *site*.

4.1.3 Sheaves on sites

The notion of a site allows us to extend the constructions above based on open sets in a topological space to more general categories. Given a site $\mathfrak{g}$, a presheaf on $\mathfrak{g}$ is a contravariant functor $\mathscr{F}$ from $\mathfrak{g}$ into the category of sets (or Abelian groups or rings etc). As before, for every object U of $\mathfrak{g}$ the elements of $\mathscr{F}(U)$ are called the sections of $\mathscr{F}$ over U. In particular if X is a final object in $\mathfrak{g}$ (i.e. there is a morphism from every object to X, so if $\mathfrak{g}$ was the set of open subsets of a topological space X with inclusion as the morphisms, X would be the final object in $\mathfrak{g}$) then the sections over X are called *global sections.*

For every morphism $V \to U$ and for every $s \in \mathscr{F}(U)$ one writes $s_{|V}$ (s restricted to V) for the image of s in $\mathscr{F}(V)$.

If $\underline{\mathfrak{U}}$ is a sieve on U, a section is given $\underline{\mathfrak{U}}$-locally if, for all $V \to U$ belonging to $\underline{\mathfrak{U}}$, there is a section $s_V \in \mathscr{F}(V)$ such that for every morphism $W \to V$ one has $s_{V|W} = s_W$.

$\mathscr{F}$ is a sheaf if, for every object U of $\mathfrak{g}$, for each sieve $\underline{\mathfrak{U}}$ covering U and every $\underline{\mathfrak{U}}$-local section $\{s_V\}$, there exists a unique section $s \in \mathscr{F}(U)$ such that $s_{|V} = s_V$, for every $V \to U$ belonging to $\underline{\mathfrak{U}}$.

Explicitly, if $\{U_i \to U\}$ is a covering then $\mathscr{F}$ is a sheaf if

$$\mathscr{F}(U) \to \coprod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \coprod_{(i,j) \in I \times I} \mathscr{F}(U_i \times_U U_j)$$

is exact.

4.2 Étale morphisms

We want now to apply some of the above constructions to the case of the étale topology of a scheme. To do this we want the site to comprise schemes over a base scheme where the morphisms have some additional properties.

Let $f : A \to B$ be a ring homomorphism, then f is flat if B is a flat A-module. (B is an A-module by restriction of scalars and is flat if $- \otimes_A B$ is an exact functor on exact sequences of A-modules).

If $f : A \to B$ is a local homomorphism of local rings then f is *unramified* if

1. the extension of the maximal ideal in A is the maximal ideal in B (i.e. $f(\mathfrak{m}_A)B = \mathfrak{m}_B$) and
2. the residue field of B is a finite separable field extension of the residue field of A.

A morphism of schemes $f : Y \to X$ is of *finite type* if for each affine scheme $V(=$ Spec$(A))$ in a cover of X, $f^{-1}(V)$ is covered by finitely many affine schemes $U(=$ Spec$(B))$ such that each B is a finitely generated A-algebra.

A morphism of schemes $f : Y \to X$ is *étale* if it is of finite type and the corresponding local homomorphisms

$$f^{\#}_{\mathsf{P}} : \mathscr{O}_{X,f(\mathsf{P})} \to \mathscr{O}_{Y,\mathsf{P}}$$

are flat and unramified at every point P.

Note that if $f : X \to S$ and $g : Y \to S$ are étale then each S-morphism from X to Y is étale. (The flatness of the morphism means that the fibres of f have the same dimension which is dim X $-$ dim Y).

4.2.1 Example of étale morphism

Suppose $X$is Spec(k) for some field k.

$$f : X \to Y \text{ unramified } \Rightarrow Y = \coprod \text{Spec}(K_i)$$

where each K_i is a finite separable field extension of k. So, for example,

$$\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})$$

is the only non-trivial étale morphism from a connected scheme to Spec$(\mathbb{R})$.

4.3 Étale topology and site

If X is a scheme and $\mathfrak{g}$ is the category of étale schemes over X then as noted above the morphisms of $\mathfrak{g}$ are étale. An *étale topology* on $\mathfrak{g}$ is a Grothendieck topology where a sieve on U is covering if it is generated by a finite family of (étale) morphisms $\phi_i : U_i \to U$ such that the images of the ϕ_i cover U.

$$\begin{tikzcd} \{U_i\} \arrow[rr, "\{\phi_i\}"] \arrow[dr, "\{f_i\}"'] & & U \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

An étale site X_{et} of X is the site defined by $\mathfrak{g}$ with the étale topology.

4.4 Étale sheaves and cohomology

As above, a sheaf $\mathscr{F}$ on X_{et} is a contravariant functor (strictly on $\mathfrak{g}$) to an appropriate category such that

$$\mathscr{F}(U) \to \coprod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \coprod_{(i,j) \in I \times I} \mathscr{F}(U_i \times_U U_j)$$

is exact for every $U \to X$ étale and every étale covering $\{U_i \to U\}$.

4.4.1 Examples of étale sheaves on X_{et}

(These are based on Chapter 6 of Milne's notes M2, where the demonstrations that they are indeed sheaves can be found)

4.4.1.1 Structure sheaf

For any $U \to X$ étale take the obvious global section (cf 2.3 above):-

$$\mathscr{O}_{X_{et}}(U) = \Gamma(U, \mathscr{O}_U)$$

4.4.1.2 Sheaf defined by a scheme

If Z is a scheme over X and $U \to X$ étale, then

$$\mathscr{F}(U) = \mathrm{Hom}_X(U, Z)$$

is a sheaf of sets. In particular:

1. If μ_n is the scheme (variety) defined by the equation $T^n - 1 = 0$ then $\mu_n(U)$ is the group of n^{th} roots of unity in $\Gamma(U, \mathscr{O}_U)$.
2. Similarly, we have the multiplicative group sheaf $\mathbb{G}_m(U) = \Gamma(U, \mathscr{O}_U)^*$.

One then has

$$H^0(X, \mathbb{G}_m) = H^0(X, \mathscr{O}_X)^*$$

In particular if X is a nice scheme (reduced, connected and proper) over an algebraically closed field k then one has

$$H^0(X, \mathbb{G}_m) = k^*.$$

4.4.2 Étale cohomology

As noted above, the category of sheaves of Abelian groups is an Abelian category with enough injectives. In particular this applies to the category of sheaves $\mathfrak{Sh}(X_{et})$ on an étale site. Thus from the functor

$$\mathscr{F} \to \Gamma(X, \mathscr{F}) : \mathfrak{Sh}(X_{et}) \to \mathfrak{Ab}$$

We can define $H^q(X_{et}, -)$ to be its q^{th} derived functor.

The theory of derived functors gives the following (amongst other things):

1. $H^0(X_{et}, \mathscr{F}) = \Gamma(X, \mathscr{F})$.
2. A short exact sequence of sheaves gives a long exact sequence of cohomology.

In addition, the resulting cohomology theory can be shown to satisfy the Eilenberg and Steenrod axioms of dimension, exactness, excision and homotopy (see Milne, M2 Chapter 9).

4.5 Comparision theorems

There are rather complex demonstrations that étale cohomology and simplicial cohomology agree under suitable circumstances. However, we will just need the following result:

For $X = \text{Spec}(K)$, the spectrum of a field, étale cohomology is identified with the Galois cohomology.

4.6 What points mean

Where a ring A is the polynomial ring of an algebraically closed field then the maximal ideals correspond to *closed points* in the affine space and the prime ideals correspond to subvarieties.

In Spec(A) the maximal ideals are closed points but the prime ideals are a different sort of point whose closure corresponds to all their closed points and subvarieties. Thus for example the closure of the zero ideal, being all of Spec(A), suggests that this point "touches" all the other points. Thus such points are called *generic points*. The result is that the topology on Spec(A) is somewhat richer than that defined for an algebraic set.

More precisely, if Z is an irreducible closed subset of Spec(A), then a point z $\in Z$ is a generic point of Z if Z is the closure of z (every open subset of Z contains z). In particular if $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ in A, then $\mathfrak{p} \in$ Spec(A) is its (unique) generic point (Mumford, M5 Chapter II, Proposition 1).

When we turn to schemes, and in particular when we want to consider neighbourhoods in an étale topology we need a more general idea. A *geometric point* of a scheme X is a morphism from Spec(k) to X where k is a separably closed field. If X is a variety (thus defined over some algebraically closed field k) then the geometric points are identified with the closed points. For an étale scheme we need to specify how a geometric point is embedded in a neighbourhood as well as in the scheme:

An étale neighbourhood of a geometric point x of X is a pair (U, u) where $U \to X$ is étale and x $= U \circ u$. i.e. we have

$$\begin{tikzcd} & U \arrow[d] \\ \text{Spec}(k) \arrow[ur, "u"] \arrow[r, "\mathrm{x}"'] & X \end{tikzcd}$$

With a little work, this can be used to extend the idea of a stalk. If $\mathscr{F}$ is a sheaf on X_{et} and x is a geometric point of X then we can define the stalk

$$\mathscr{F}_X = \varinjlim \mathscr{F}(U)$$

Where the limit is taken over all étale neighbourhoods of x.

(This construction is not used explicitly in this essay, although it is used in the proof of some of the results referenced).

§5. Preliminary results

This section introduces some results that will be needed when we come to look at the cohomology of curves.

The Brauer group of a field is related to a Galois cohomology group and its vanishing under certain circumstances gives us a corresponding result in cohomology. Some time is spent defining central simple algebras used in building the group as these ideas may not be familiar to those (like the author) who are not well versed in the theory of Algebraic Groups. A sketch is given of the demonstration of the relationship with cohomology together with a statement of Tsen's theorem which leads to the vanishing of the group.

Some basic facts about the Kummer sequence and the Picard group are stated for later use. (The equivalence of the definitions of the Picard group in terms of invertible sheaves and in terms of divisors and the relationship between Weil divisors and Cartier divisors in the case of curves has, perforce, been assumed for the purposes of this essay).

5.1 Brauer Group and Galois cohomology

5.1.1 Construction of the Brauer group

Recall (Milne, M3 Chapter IV section 2) If A and B are finite dimensional k-algebras (k a field):

1. $A \otimes_k B$ is also a k-algebra.

   In particular if $Z(A)$ is the centre of A then

   $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$.

   Note that the tensor product is commutative in the sense that there is a unique isomorphism $A \otimes_k B \to B \otimes_k A$ sending $a \otimes b$ to $b \otimes a$

   and associative in the sense that there is a unique isomorphism

   $A \otimes_k (B \otimes_k C) \to (A \otimes_k B) \otimes_k C$ sending $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$.

2. A is *simple* if it has no non-trivial two-sided ideals.

   Then Wedderburn's theorem tells us that A is k-isomorphic to $M_n(D)$ for some division algebra D (the algebra of square matrices over D).

3. If the centre of A is precisely k then A is a *central simple algebra* (CSA).

   In particular, if A is a CSA then the division algebra D above is also central since $Z(M_n(k) \cong k$ and $M_n(D) \cong M_n(k) \otimes_k D$ gives us that

   $k = Z(A) \cong Z(M_n(D)) \cong k \otimes_k Z(D) \cong Z(D)$.

4. If A and B are CSA then $A \otimes_k B$ is also CSA (M3, Corollary 2.8).

5. A^{op} is defined by reversing the order of multiplication in A. Then if A is a CSA

   $A \otimes_k A^{op} \cong M_n(k)$ where $n = [A : k]$ (M3, Corollary 2.9).

6. Note also that $M_m(k) \otimes_k M_n(k) = M_{mn}(k)$ (M3, Example 2.2).

If A and B are CSAs over a field k, we can define an equivalence relation: $A \sim B$ if the corresponding division algebras in 2. above are isomorphic.

Clearly $M_n(k) \sim M_m(k) \; \forall \; n, m$.

The group operation on classes is given by $[A].[B] = [A \otimes_k B]$ which is a CSA by 4.

The identity element is $[M_m(k)]$ which follows from 6. above, since if $A \cong M_n(D)$,

$M_m(k) \otimes_k A \cong M_m(k) \otimes_k M_n(D) \cong (M_m(k) \otimes_k M_n(k)) \otimes_k D$

$\cong M_{mn}(k) \otimes_k D \cong M_{mn}(D) \sim A$

The inverse of $[A]$ is $[A^{op}]$ by 5. above.

Thus the with the properties of the tensor product in 1. above, we have an (Abelian) group, denoted Br(k).

5.1.2 Field extensions

If A is a CSA over a field A and K'/K is a finite field extension then $A \otimes_K K'$ is a CSA over K'. In fact there exists an integer $n > 0$ such that $A \otimes_K K' \cong M_n(K')$ (Gille, G1 Theorem 2.2.1).

Further, for any A that is a CSA over K there is a finite Galois field extension K'/K such that $A \otimes_K K' \cong M_n(K')$ (Gille, G1 Corollary 2.2.6).

This enables us to partition Br(K) into subgroups

$$\text{Br}(n, K) = \{A \in \text{Br}(K) | A \otimes_K K' \cong M_n(K') \text{ with } K'/K \text{ Galois}\}.$$

$$\text{Then Br}(K) = \bigcup \text{Br}(n, K)$$

5.1.3 Theorem: Brauer Group of a field and cohomology

There is the following group isomorphism:

$$\text{Br}(K) \cong H^2(G, \bar{K}^*)$$

Where K is a field (with closure $\bar{K}$), $G = \text{Gal}(\bar{K}/K)$ and Br(K) is the Brauer group of K as defined above:

5.1.3.1 Proof (sketch)

Let $\bar{K}$ be an algebraic closure of K and $G = \text{Gal}(\bar{K}/K)$. Then it is a result from Galois cohomology (S1 Chapter III §1) that

$$\text{Br}(n, K) \cong H^1(G, \text{Aut}(M_n(\bar{K})))$$

Since a Galois group is a profinite group this cohomology class makes sense if G acts on Aut$(M_n(\bar{K}))$.

If $\phi \in$ Aut$(M_n(\bar{K})),\ g \in G$ and $M \in M_n(\bar{K})$, then we have an action given by $g\phi(M) = g(\phi(g^{-1}M))$ where G acts on M through its action on $\bar{K}$.

We want to identify Aut$(M_n(\bar{K}))$:-

It is a fact about matrix rings that over a field k all automorphisms of $M_n(k)$ are inner automorphisms. i.e. given by $M \to CMC^{-1}$ for some invertible matrix C (Gille, G1 Lemma 2.4.1).

i.e. the corresponding natural homomorphism $GL_n(k) \to$ Aut$(M_n(k))$ is surjective and its kernel $= k^*I$ (the subgroup of scalar matrices). Thus

$$\text{Aut}(M_n(k)) \cong GL_n(k)/k^*I \cong PGL_n(k)$$

So, specialising to $\bar{K}$, we have a short exact sequence

$$1 \to \bar{K}^* \to GL_n(\bar{K}) \to PGL_n(\bar{K}) \to 1$$

and from the corresponding long sequence in cohomology we have a map

$$H^1(G, PGL_n(\bar{K})) \to H^2(G, \bar{K}^*)$$

Or, by substituting in the above relations we have

$$\text{Br}(n, K) \to H^2(G, \bar{K}^*).$$

This can be extended to a map

$$\text{Br}(K) \to H^2(G, \bar{K}^*)$$

Which can be shown to be an isomorphism (Deligne, D1 Chapter III, Proposition(1.3))

5.1.4 Tsen's theorem and the vanishing cohomology

(The following results are provided by Deligne, D1 Chapter III).

If k is an algebraically closed field and K is an extension of transcendence degree 1 over k then Br$(K) = 0$. (D1, §2).

This leads to the following result in Galois cohomology:

$$H^q(G, K^*) = 0 \text{ for all } q > 0 \ \ \text{(D1, Proposition 1.6)}$$

and combining with the comparison theorem between étale and Galois cohomology we have that if k is an algebraically closed field and K is an extension of transcendence degree 1 over k then the étale cohomology groups $H^q(\text{Spec}(K), \mathbb{G}_m) = 0$ for all $q > 0$. (D1, Corollary 2.4).

5.2 Kummer Theory

5.2.1 Multiplicative group and roots of unity

As noted above, there is the sheaf providing the multiplicative group of units of sections of the structure sheaf

$$\mathbb{G}_m : U \to \Gamma(U, \mathscr{O}_U)^*.$$

We have a map $\mathbb{G}_m(U) \to \mathbb{G}_m(U)$ given by $x \mapsto x^n$ and the kernel of this map is $\mu_n(U)$, as also defined above (n prime to the characteristic of the base field).

If X is a scheme over a separably closed field K and if n is invertible on X, the choice of a primitive n^{th} root of unity $\xi \in K$ defines an isomorphism

$$\mathbb{Z}/n\mathbb{Z} \to \mu_n(U)$$

$$i \mapsto \xi^i$$

(This is clearly non-canonical).

5.2.2 Kummer sequence

We have an exact sequence of sheaves resulting from raising to the n^{th} power:

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

This is only exact in the étale topology.

5.3 Invertible sheaves and the Picard group

5.3.1 $\mathscr{O}_X$-modules

Let $(X, \mathscr{O}_X)$ be a ringed space. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ on X such that for each open set $U \subseteq X$, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module and restrictions are compatible with the module structures.

The tensor product $\mathscr{F} \otimes \mathscr{G}$ of two $\mathscr{O}_X$-modules is the $\mathscr{O}_X$-module given by the sections $\mathscr{F}(U) \otimes \mathscr{G}(U)$ (where tensoring is over $\mathscr{O}_X(U)$).

An $\mathscr{O}_X$-module $\mathscr{F}$ is an *invertible sheaf* (i.e. locally free of rank 1) if X can be covered by open sets U for which $\mathscr{F}_{|U}$ is an $\mathscr{O}_{X_{|U}}$-module that is isomorphic to $\mathscr{O}_{X_{|U}}$.

The invertible sheaves on a ringed space form an Abelian group (up to isomorphism) under tensor product over $\mathscr{O}_X$. This has $\mathscr{O}_X$ as the identity and $\mathscr{F}^{-1}$ is the dual $\mathscr{O}_X$-module Hom$(\mathscr{F}, \mathscr{O}_X)$.

5.3.2 Theorem: Pic(X) and cohomology

There is an isomorphism $H^1(X, \mathbb{G}_m) \cong$ Pic(X).

Where Pic(X) is the group of isomorphism classes of invertible sheaves on X. (Deligne, D1, Chapter II, Proposition 2.3).

§6. Cohomology of curves

("L'étude des courbes est la clef de la cohomologie étale" - Deligne, D1)

This section brings together many of the constructions and results above to calculate the étale cohomology for the constant sheaf $\mathbb{Z}/n\mathbb{Z}$. Results for the multiplicative group are extended via the Kummer sequence to results for the group of units which can be identified with $\mathbb{Z}/n\mathbb{Z}$. Some results about the Jacobian of a curve (and about Abelian varieties) are quoted to complete the calculations.

We will be considering a projective, non-singular, connected curve X over an algebraically closed field k.

6.1 Cohomology of $\mathbb{G}_m$

Collecting some of the above results we have

$$H^0(X, \mathbb{G}_m) = k^*$$

$$H^1(X, \mathbb{G}_m) = \text{Pic}(X).$$

And in addition (Deligne, D1 Chapter III proposition 3.1)

$$H^q(X, \mathbb{G}_m) = 0 \text{ for } q \geq 2.$$

(This is a non-trivial result. Its demonstration relies on the relationship between étale and Galois cohomology stated in the sub-section on Comparison theorems and the consequences of Tsen's theorem stated above. It also involves working with geometric points, higher direct images, spectral sequences etc. All good fun but space prohibits the re-interpretation of the proof that can be found in Deligne, D1).

6.2 Cohomology of μ_n

(In view of the identification in the section on Kummer theory, we are effectively looking at cohomology with coefficients in the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, which is an important step along the way towards l-adic cohomology, sadly not considered in this essay).

If we start from the Kummer sequence (where n is prime to the characteristic of k)

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

We have the corresponding long exact sequence in étale cohomology

$$0 \to H^0(X,\mu_n) \to H^0(X,\mathbb{G}_m) \xrightarrow{n} H^0(X,\mathbb{G}_m) \to H^1(X,\mu_n) \to H^1(X,\mathbb{G}_m) \xrightarrow{n} H^1(X,\mathbb{G}_m) \to H^2(X,\mu_n) \to 0$$

Using the fact that $H^2(X, \mathbb{G}_m) = 0$.

On the left we have that $H^0(X, \mu_n) \to k^* \to k^*$ using the fact that $H^0(X, \mathbb{G}_m) = k^*$. i.e. $H^0(X, \mu_n) = \mu_n(X)$.

On the right we have the following exact sequence to unpick:

$$H^1(X, \mu_n) \to \text{Pic}(X) \xrightarrow{n} \text{Pic}(X) \to H^2(X, \mu_n) \to 0.$$

For a curve, Pic(X) also has a manifestation as the divisor class group

$$\text{Pic}(X) = \frac{\text{Divisors}}{\text{Principal divisors}}$$

and in particular has a subgroup Pic$^0(X)$ corresponding to divisors of degree 0. (From Milne, M2 Proposition 14.1) we have an exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0$$

So the torsion is in Pic$^0(X)$. Now Pic$^0(X)$ can be identified with $J(k)$ where J is the Jacobian variety of the curve X which has dimension $g = \text{genus}(X)$. As J is an Abelian variety, the map

$$n : J(k) \to J(k)$$

is surjective with kernel $(\mathbb{Z}/n\mathbb{Z})^2$. (Milne, M4 Theorem 8.2). i.e. we have

$$H^1(X, \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

Finally from the end of the cohomology exact sequence we have

$$H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

and $H^q(X, \mathbb{Z}/n\mathbb{Z}) = 0$ for $q > 2$ (since we can identify μ_n with $\mathbb{Z}/n\mathbb{Z}$).

§7. Conclusion

This essay has been a fairly elementary excursion into the realms of étale cohomology which has required the introduction of a lot of definitions. It has set out the basic concepts (with some supporting background material) and collected together some results that lead up to the calculations of the étale cohomology of well behaved curves with coefficients in a constant sheaf $\mathbb{Z}/n\mathbb{Z}$. This has depended upon the Kummer sequence being exact which is one of the consequences of the étale theory. In short, we have indicated how the following results may be obtained for a (projective, non-singular, connected) curve X of genus g over an algebraically closed field :

$$H^0(X, \mathbb{Z}/n\mathbb{Z}) = \mu_n$$
$$H^1(X, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{2g}$$
$$H^2(X, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$$
$$H^q(X, \mathbb{Z}/n\mathbb{Z}) = 0 \text{ for } q > 2$$

There are a number of technical results needed to make the theory respectable and to demonstrate that it can properly take its place amongst other cohomology theories, but these have been largely assumed.

The next step towards achieving significant progress (and in particular tackling the Weil conjectures) would need the additional machinery of l-adic methods. Next time!

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